Determining the point-equivalent symmetries of initial-value problems

Richard Watts[*] Peter E Hydon[†]

Abstract

Although symmetry analysis provides a powerful tool for solving differential equations, it has not proved to be so successful in the treatment of initial- and boundary-value problems. A possible reason for this is the belief that the set of symmetries of an initial-value problem is a subset of the symmetries of the governing differential equation. It was recently shown that this is, in fact, not the case and a method was introduced for constructing the symmetries of a class of initial-value problems using Taylor series. We extend this method for arbitrary-order regular ordinary differential equations subject to both an arbitrary-order single initial condition and an arbitrary linear combination of initial conditions. Furthermore, we propose a practical method for dealing with a class of IVPs that possess a regular singularity through the use of the Frobenius method.

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1 Introduction

Toward the end of the 19th century, the Norwegian mathematician Sophus Lie developed an astonishing theory that provided a fruitful, systematic mechanism for solving differential equations. Although Lie's approach and its many extensions have had great success in determining and classifying solutions of differential equations [10], it has proved much less successful in the treatment of initial-value problems and boundary-value problems (IVPs and BVPs respectively). Indeed, the existence and determination of a "general procedure for applying symmetry methods to BVPs" is considered one of the significant open problems in the area of symmetry analysis [4].

It was believed [3] that a one-parameter Lie group of transformations is admitted by a BVP if and only if it leaves invariant the

1. boundary

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[*] Email: R.Watts@surrey.ac.uk

[†] Email: P.Hydon@surrey.ac.uk

2. boundary conditions and
3. governing differential equation.

As we shall see, however, these conditions do not allow a user to find the full Lie group of symmetries of a given problem. Specifically, condition (3) is too restrictive, since a user is not interested in all solutions of a given differential equation but only those that also satisfy the initial conditions.

Under the assumption of points (1)-(3), research efforts have developed specific approaches that do not employ reductive algorithms reminiscent of Lie's work. Instead, methods have concentrated on the alternative problem of classifying initial conditions which will, through the application of familiar symmetry methods, solve classes of associated IVPs. For example, some IVPs can be solved through the use of an iterative approach [6]. Starting with a given IVP and armed with knowledge of (possibly trivial) solutions of the governing PDE, the method uses the infinite-dimensional part of the Lie algebra to build a hierarchy of associated IVPs until one is found that can be solved. Once armed with this solution, a user can work back along the chain to find the symmetries of the original IVP. Other approaches, under the assumption of points (1)-(3) believe classical symmetries are insufficiently rich to aid the solution of IVPs, and employ higher-order and nonclassical symmetry techniques instead (see any of [2, 5, 16, 17]). Such techniques show that higher-order conditional symmetries are responsible for the reduction of IVPs to Cauchy problems for some system of differential equations, under appropriate ansatze concerning the structure of solutions of the invariant surface condition. Thus, a user can classify those IVPs that can be reduced to Cauchy problems, a classification that is performed for evolution equations in two independent variables.

Utilising Taylor series, Hydon [9] recently introduced a method to find the symmetry generators of a class of IVPs,

$$y''' = \omega(x, y, y', y'') \quad \text{subject to} \quad y''(0) = 0, \tag{1}$$

where ω is a regular polynomial in x and y''. The purpose of the current paper is to extend the method of Hydon (herein referred to as **H**) to arbitrary-order ODEs subject to a linear combination of arbitrary-order initial conditions, incorporating those associated reduced equations which may contain a regular singularity at the point $x = 0$.

In §2, we provide a brief explanation detailing why point (3) is not required and state, in full generality, the theorem which allows a user to construct the symmetry generators of a given IVP. We illustrate examples of this theorem in §3 and provide a full proof in §4. Finally, we provide some suggestions for possible extensions to the work in the last section.

2 Reasoning and statement of the theorem

It is apposite to provide a brief motivation for the theorem which concludes this section. Recall that

$$X = Q(x, y, y')\partial_y$$

generates dynamical symmetries of a given nth order ordinary differential equation (ODE)

$$y^{(n)} = \omega(x, y, y', \ldots, y^{(n-1)}) \tag{2}$$

if

$$Q(x, y, y') = \eta(x, y) - y'\xi(x, y) \tag{3}$$

is a solution of the linearised symmetry condition (henceforth LSC)

$$\Gamma = D^n Q - \sum_{k=0}^{\infty} \omega_{y^{(k)}} D^k Q = 0, \tag{4}$$

where D is the total derivative on solutions of (2) given by

$$D = \partial_x + y'\partial_y + \cdots + \omega\partial_{y^{(n-1)}}. \tag{5}$$

In particular, we restrict attention to those dynamical symmetries such that the characteristic Q is linear in y'; such symmetries will be referred to as point-equivalent symmetries. To find such solutions $Q(x, y, y')$, note that (4) splits into an over-determined system of PDEs for $\xi(x, y)$ and $\eta(x, y)$, since these functions are independent of derivatives of y, allowing a user to find (3) and hence the symmetries of (2). See any one of [3, 8, 11, 13, 14] for a modern exposition.

For example, the simplest third-order ODE

$$y''' = 0, \tag{6}$$

has a 7-dimensional Lie algebra of point-equivalent symmetry generators, spanned by

$$X_1 = \partial_y, \quad X_2 = y\partial_y, \quad X_3 = x^2\partial_y, \quad X_4 = xy'\partial_y, \quad X_5 = y'\partial_y,$$
$$X_6 = x\partial_y, \quad X_7 = (2xy - x^2y')\partial_y.$$

Note that the first integrals of (6) are given by

$$\alpha = y - xy' + \frac{1}{2}x^2y'', \quad \beta = y' - xy'', \quad \gamma = y''. \tag{7}$$

(Recall that a non-constant function $\alpha(x, y, y', y'')$ is a first integral of an ODE if and only if it is constant along solutions of the ODE i.e. if and only if $D\alpha = 0$).

If we impose the initial condition

$$y''(0) = 0$$

on (6), the reduced equation is that equation for which $\gamma = 0$ in (7), namely

$$y'' = 0.$$

This equation has an 8-dimensional Lie algebra of point-equivalent symmetry generators, spanned by

$$\begin{gathered} X_1 = \partial_y, \quad X_2 = y\partial_y, \quad X_3 = x^2\partial_y, \quad X_4 = xy'\partial_y, \quad X_5 = y'\partial_y, \\ \bar{X}_6 = yy'\partial_y, \quad \bar{X}_7 = (xy - x^2y')\partial_y, \quad \bar{X}_8 = (y^2 - xyy')\partial_y. \end{gathered} \tag{8}$$

Similarly, if we impose the initial condition $y'(0) = 0$ on (6), the reduced equation is given by $\beta = 0$, or

$$y'' = \frac{y'}{x}, \tag{9}$$

which has the 7-dimensional Lie algebra spanned by

$$\begin{gathered} X_1 = \partial_y, \quad X_2 = y\partial_y, \quad X_3 = x^2\partial_y, \quad X_4 = -xy'\partial_y, \\ \tilde{X}_5 = (2y^2 - xyy')\partial_y, \quad \tilde{X}_6 = (2x^2y - x^3y')\partial_y, \quad \tilde{X}_7 = \tfrac{y'}{x}\partial_y, \quad \tilde{X}_8 = \tfrac{yy'}{x}\partial_y. \end{gathered} \tag{10}$$

Clearly, the symmetries of the respective reduced equations do not coincide with those of the governing equation (6), contrary to point (3) in §1 and first noted in [9]. In particular, the symmetry generators (8) and (10) coincide with the symmetries of (6), whilst $\bar{X}_6 - \bar{X}_8$ and $\tilde{X}_5 - \tilde{X}_8$ are new.

It is convenient to work in the space of first integrals to see both why these new symmetries are not symmetries of the original equation (6) and why some symmetries of the original equation are note symmetries of the reduced equations. In particular, we shall concentrate on the example provided by the reduced equation (9).

In the space of first integrals, symmetry generators of the submanifold defined by $\beta = 0$ in (7) must leave that submanifold invariant. However, the generators $X_5 - X_8$ of the governing equation do not. For example, in the space of first integrals, X_5 becomes (through application of the chain rule)

$$X_5 = \beta\partial_\alpha + \gamma\partial_\beta,$$

which generates translations in the β-direction depending on the value of γ when $\beta = 0$. Thus, it is not a symmetry of (9). Similarly, X_6 and X_7 are not symmetries of (9) either.

Conversely, it becomes obvious why the new generators $\tilde{X}_5 - \tilde{X}_8$ are symmetries of (9). In the space of first integrals, dynamical symmetries act as point transformations; thus, they are independent of x [8]. Initially, $\tilde{X}_5 - \tilde{X}_8$ depend upon x, which explains why they are not symmetries of the original equation; for example

$$X_7 = \left(\frac{3\beta}{x} + \gamma\right)\partial_\alpha - 3\frac{\beta}{x^2}\partial_\beta + 2\frac{\beta}{x^3}\partial_\gamma.$$

On the submanifold $\beta = 0$, however,

$$X_7 = \gamma \partial_\alpha,$$

which is not only independent of x, but also leaves $\beta = 0$ invariant. Thus, it is a symmetry of (9), as are $\tilde{X}_5$, $\tilde{X}_6$ and $\tilde{X}_8$.

According to **H**, a user can find the symmetries of the IVP (1) by solving the determining system

$$D^2 Q|_0 = 0 \quad \text{and} \quad (D^k \Gamma)|_0 = 0 \quad \forall\, k \in \mathbb{N}_0,$$

where Γ is given by (4) (with $n = 3$), Q is written as a Taylor series about the point $x = 0$ as follows:

$$Q(x, y, y') = \sum_{k=0}^{\infty} \frac{x^k}{k!} \Big(\eta_k(y) - y' \xi_k(y) \Big) \tag{11}$$

and the notation $|_0$ denotes setting

$$x = 0, \quad y = \alpha, \quad y' = \beta, \quad y'' = 0.$$

The determining system is valid for all third-order ODEs where ω is a regular polynomial in x and y''.

We now state, in full generality, the theorem which extends **H** and provides a user with the necessary determining system to find the symmetries of an arbitrary-order IVP subject to an arbitrary linear combination of initial conditions.

Theorem 2.1. *For the initial-value problem*

$$y^{(n)} = \omega(x, y, \dots, y^{(n-1)})$$

subject to the linear combination of arbitrary-order initial conditions

$$y^{(m)}(0) + c_m y^{(m-1)}(0) + \cdots + c_2 y'(0) + c_1 y(0) = 0,$$

where $1 \leq m \leq n - 1$*,* $c_i \in \mathbb{Q}$*, the point-equivalent symmetry generators* $X = Q(x, y, y')\partial_y$ *are found by solving the following determining system:*

$$\Big(D^m Q + c_m (D^{m-1} Q) + \cdots + c_2 (DQ) + c_1 Q \Big)\Big|_0 = 0 \tag{12}$$

and

$$D^k \Gamma\Big|_0 = 0 \quad \forall k \in \mathbb{N}_0,$$

where Γ *is the linearised symmetry condition given by* (4) *and* Q *is given by* (11)*.*

Here, (12) expresses the invariance of the initial condition under the symmetries of the IVP.

Using a slight modification of the structure of Q in (11) (discussed in §4), Theorem 2.1 allows a user to deal with those reduced equations containing regular singularities at the point $x = 0$, as in the example of the reduced equation (9).

Since Theorem 2.1 involves the use of Taylor series about $x = 0$, it necessarily requires ω to be a regular polynomial in x and the leading-order derivative of the initial condition. It also requires that $y(x)$ is analytic at $x = 0$ so that we can write it as a Taylor series.

We will prove Theorem 2.1 is §4. In order to highlight the usefulness of this theorem, we first provide some examples in the following section.

3 Examples

In what follows, we will adopt the notation of **H** throughout, namely,

$$\xi_k(\alpha) = \left.\frac{\partial^k \xi(x,\alpha)}{\partial x^k}\right|_{x=0}, \quad \eta_k(\alpha) = \left.\frac{\partial^k \eta(x,\alpha)}{\partial x^k}\right|_{x=0}, \quad k \in \mathbb{N}_0,$$

with derivatives of $\xi_k(\alpha)$ and $\eta_k(\alpha)$ with respect to α denoted by ξ_k', ξ_k'', η_k' etc.; furthermore, the argument α will, in general, be left out.

The third-order differential equation

$$y''' = xyy'' + 2yy' + xy'^2 \tag{13}$$

has a one-dimensional Lie algebra of point-equivalent symmetry generators spanned by

$$X_1 = \left(y + \frac{1}{2}xy'\right)\partial_y.$$

To find the symmetries of the set of solutions that satisfy (13) subject to $y'(0) = 0$, we are required to solve the determining system of Theorem 2.1 for $n = 3$ and $m = 1$. In fact, we know that $Q|_0$ will take a particular form in this case (see §4), given by

$$Q(x,y,\alpha,\gamma) = \sum_{k=0}^{\infty} \eta_k(y)x^k - \sum_{k=0}^{\infty}\sum_{r=0}^{\infty} w_r(\alpha,\gamma)\xi_k(y)x^{r+k},$$

where the w_r are given by the series expansion of

$$\begin{aligned} \frac{y'(x)}{x} &= y''(0) + xD\omega|_0 + x^2(D^2\omega)|_0 + \cdots \\ &= \sum_{r=0}^{\infty} w_r(\alpha,\gamma)x^r. \end{aligned}$$

Specifically, for (13),

$$w_0 = \gamma, \quad w_2 = \frac{1}{2}\alpha\gamma \quad \text{etc.}, \qquad w_{2k+1} = 0 \,\forall\, k \in \mathbb{N}_0.$$

Hence, the invariance of the initial condition $y'(0) = 0$ under the symmetries of the IVP is given by

$$(DQ)|_0 = \eta_1 - \gamma\xi_1 = 0$$

which tells us that

$$\eta_1 = 0, \qquad \xi_1 = 0.$$

Collecting terms by powers of γ in the next determining equation gives

$$(\Gamma)|_0 = -3\eta_1'\gamma^2 + \gamma(-\xi_1\alpha + 3\eta_1' - 6\xi_3) + 6\eta_3 - 2\eta_1\alpha = 0,$$

which provides us with $\eta_3 = \xi_3 = 0$. The following determining equation $(D\Gamma)|_0 = 0$ splits to give

$$\begin{aligned} -3\xi_0'' &= 0, \\ 3\eta_0'' - 6\xi_0'\alpha - 12\xi_2' &= 0, \\ -6\xi_2\alpha - 24\xi_4 + 12\eta_2' - 3\eta_0 &= 0, \\ 24\eta_4 - 6\eta_2\alpha &= 0, \end{aligned}$$

from which

$$\xi_0 = c_2\alpha + c_1. \tag{14}$$

Continuing in this fashion (with the aid of MAPLE [15]) we see that the remaining determining equations in fact constrain (14) and that the general solution of the determining equations is

$$\xi_0 = c_1, \qquad \xi_2 = \frac{1}{2}c_3, \qquad \eta_0 = c_3\alpha$$

with

$$\xi_1 = \eta_1 = \eta_2 = 0 \;\text{ and }\; \eta_k = \xi_k = 0 \;\; \forall\, k \geq 3.$$

We can then reconstruct the characteristic $Q(x, y, y')$ to give

$$Q(x, y, y') = \frac{c_1}{x} + \frac{1}{2}c_3x + c_3y$$

such that the point-equivalent symmetry generators are given by

$$X_1 = \left(y + \frac{1}{2}xy'\right)\partial_y, \qquad X_2 = \frac{y'}{x}\partial_y. \tag{15}$$

Whilst X_1 is a symmetry of the governing equation, X_2 is a symmetry only of the reduced equation; furthermore it is of the structure accommodated only by Theorem 2.1. We can now use X_2 to find the reduced equation as follows: write X_2 in the form

$$\tilde{X}_2 = \frac{1}{x}\partial_x,$$

which has fundamental differential invariants

$$r(x,y) = y, \quad v(x,y,y') = \frac{x}{y'}.$$

Then $\tilde{X}_2$ generates point symmetries of the unknown reduced equation

$$y'' = \tilde{F}(x,y,y'), \tag{16}$$

with total derivative

$$\tilde{D}_0 = \partial_x + y'\partial_y + \tilde{F}\partial_{y'}.$$

The theory of differential invariants allows us to conclude that there exists a function $g(r,v)$ such that (16) is equivalent to the first-order ODE given by

$$\frac{\mathrm{d}v}{\mathrm{d}r} = g(r,v).$$

Thus

$$\tilde{F} = \frac{g(r,v)(r_x + y'r_y) - v_x - y'v_y}{v_{y'}}, \tag{17}$$

which we can substitute into the identity

$$\tilde{D}_0\tilde{F} = \omega(x,y,y',\tilde{F}),$$

where ω is given by (13), to find the unknown function $g(r,v)$, and hence the reduced equation $\tilde{F}$ from (17). Hence,

$$\tilde{F} = \frac{y'}{x}\Big(1 - y'^2 g(r,v)\Big),$$

which leads to the identity

$$3x\frac{r}{v} - x^3\frac{gr}{v^3} + x^3\frac{1}{v^2} = -3x\frac{g}{v^3} - x^3\frac{g_r}{v^4} + 3x^3\frac{g^2}{v^5} - x^3\frac{gg_v}{v^4}.$$

By looking at powers of x, we obtain

$$g(r,v) = -rv^2 = -\frac{x^2y}{y'^2}$$

and so the reduced equation, from (17), is

$$y'' = \frac{y'}{x}(1 + x^2y). \tag{18}$$

The reduced equation (18) has the two symmetries (15), which allow a user to find the solution to (18) and hence the solution to the original IVP.

The second example considers the third-order ODE

$$y''' = \frac{1}{y^2} \tag{19}$$

subject to the linear combination of initial conditions

$$y''(0) + ay'(0) + by(0) = 0. \tag{20}$$

The governing equation (19) has symmetries generated by

$$\ldots$$

Using the rifsimp [Reference?] package in MAPLE, a solution was found subject to the constraints

$$a^2 - 2b = 0, \quad a \neq 0,$$

the solution being

$$\eta_0 = -\frac{c_1 y}{a}, \quad \eta_1 = c_1 y \quad \xi_1 = -\frac{c_1}{a}, \quad \eta_2 = c_1$$

and all other η_k, ξ_k zero. This provides us with the point-equivalent symmetry generator

$$X_1 = x\left(x - \frac{1}{a}\right)\partial_x + y\left(x - \frac{1}{a}\right)\partial_y.$$

Note that if $a = b = 0$, i.e. we have the IVP given by (19) subject to $y''(0) = 0$, we have

$$\tilde{X}_1 = x\partial_x + y\partial_y.$$

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4 Proof of the theorem

Now that we have seen an illustration of Theorem 2.1 in action, we provide its proof. First, we provide a quick sketch of how we will go about this. Following **H**, the main idea is to find a relationship between the LSCs of the reduced and governing equations respectively i.e. we want to write things we don't know in terms of those we do. To do this, we display an equivalence between the respective total derivatives on solutions of the reduced equation and exploit the consequences. Once this is done, under the assumption that the LSC of the reduced equation can be written as a Taylor series, we can write down the determining system stated in Theorem 2.1.

Proof. We will proceed in two parts: first, we will prove a restricted form of the theorem in order to become familiar with the steps required and then generalise this result to complete the proof. Let us introduce some notation: the LSC for the governing ODE (2) is (4), with D defined by (5). Imposition on (2) of the initial condition

$$y^{(m)}(0) = 0, \quad 1 \leq m \leq n-2,$$

results in the reduced equation

$$y^{(n-1)} = \tilde{F}(x, y, \ldots, y^{(n-2)}). \tag{21}$$

The LSC for (21) is $\tilde{\Gamma}_0 = 0$, where

$$\tilde{\Gamma}_0 = \tilde{D}_0 Q - \sum_{k=0}^{n-2} \tilde{F}_{y^{(k)}} \left(\tilde{D}_0^k Q \right) \tag{22}$$

and $\tilde{D}_0$ is the total derivative on solutions of (21) given by

$$\tilde{D}_0 = \partial_x + y' \partial_y + \cdots + F \partial_{y^{(n-2)}}.$$

We cannot proceed with the problem in this form, since (21) possibly contravenes the requirement to be able to write (22) as a Taylor series about $x = 0$ (see, for example, the reduced equation (9) in §2, which contains a regular singularity at $x = 0$). Instead, we adjust $\mathbf{H}$ and solve the reduced equation for the mth derivative, writing the reduced equation as

$$y^{(m)} = F(x, y, \ldots, y^{(m-1)}, y^{(m+1)}, \ldots, y^{(n-1)}). \tag{23}$$

Now, F no longer has a singularity since, by definition,

$$y^{(m)} = F(0, \ldots, y^{(n-1)}(0)) \equiv 0.$$

The total derivative is replaced by

$$D_0 = \partial_x + \cdots + F \partial_{y^{(m-1)}} + y^{(m+2)} \partial_{y^{(m+1)}} + \cdots + \omega|_{RE} \partial_{y^{(n-1)}}, \tag{24}$$

where RE represents (23) i.e.

$$\omega|_{RE} = \omega(x, \ldots, y^{(m-1)}, F, y^{(m+1)}, \ldots, y^{(n-1)})$$

and the LSC becomes $\Gamma_0 = 0$, where

$$\Gamma_0 = D_0^m Q - \sum_{k=m+1}^{n-1} F_{y^{(k)}} (D_0^k Q) - \sum_{k=0}^{m-1} F_{y^{(k)}} (D_0^k Q). \tag{25}$$

We can now proceed in confidence, knowing that we can write (25) as Taylor series about $x = 0$. Given an arbitrary differentiable function $H(x, y, \ldots, y^{(n-1)})$, define

$$h(x, \ldots, y^{(m-1)}, y^{(m+1)}, \ldots, y^{(n-1)}) = H(x, \ldots, F, \ldots, y^{(n-1)}).$$

Using the crucial identity

$$D_0 F \equiv y^{(m+1)} \tag{26}$$

and the chain rule, it is straightforward to show that

$$D_0 h(x, \ldots, y^{(m-1)}, y^{(m+1)}, \ldots, y^{(n-1)}) = \Big(DH(x, y, \ldots, y^{(n-1)}) \Big)|_{RE},$$

and, by induction,

$$D_0^k h(x, \ldots, y^{(m-1)}, y^{(m+1)}, \ldots, y^{(n-1)}) = \Big(D^k H(x, y, \ldots, y^{(n-1)}) \Big)|_{RE} \quad \forall\, k \in \mathbb{N}_0. \tag{27}$$

Which is to say D and D_0 are equivalent on solutions of the reduced equation.

Using these results, it is possible to prove the following relationship between (25) and (4) restricted to solutions of the reduced equation:

$$\Gamma|_{RE} = -\frac{1}{F_{y^{(n-1)}}} D_0\Gamma_0 - \left(\frac{F_{y^{(m-1)}}}{F_{y^{(n-1)}}} + (\omega_{y'})|_{RE}\right)\Gamma_0 \tag{28}$$

and

$$(D^k\Gamma)\big|_{RE} = D_0^k\left\{-\frac{1}{F_{y^{(n-1)}}} D_0\Gamma_0 - \left(\frac{F_{y^{(m-1)}}}{F_{y^{(n-1)}}} + (\omega_{y'})|_{RE}\right)\Gamma_0\right\} \tag{29}$$

for each $k \in \mathbb{N}$. To prove this result, note the crucial identity (26) and differentiate it with respect to all derivatives of $y^{(k)}$, $k \neq m$, in order to find expressions for the $\omega_{y^{(k)}}$ to substitute into (4) (remembering the equivalence of D and D_0 on the reduced equation). This proves (28); to complete the proof, apply (27) to (28).

Now that we have a relationship between (25) and (4) restricted to solutions of the reduced equation, we can complete the proof for this simpler case as follows: write (25) as a Taylor series about the point $x = 0$:

$$\Gamma_0 = \sum_{k=0}^{\infty} \frac{x^k}{k!}\left(D_0^k\Gamma_0\right)\Big|_0 = 0 \qquad \forall\, k \in \mathbb{N}_0.$$

Note that it would not be possible to perform this important step if we continued using the reduced equation in the form (21). When $k = 0$, we have

$$\Gamma_0\big|_0 = 0. \tag{30}$$

Recall that the LSC for the reduced equation is given by (25). About the point $x = 0$, noting that $F(0, \alpha_0, \gamma_0) \equiv 0$ and $Q\big|_0 \equiv Q_0$, and recalling the induction result(27), we see that (30) becomes

$$\Gamma_0\big|_0 = (D_0^m Q)\big|_0 = (D^m Q)\big|_0 = 0.$$

For all other $k \geq 0$, (29) allows us to conclude that

$$(D^k\Gamma)\big|_0 = 0 \quad \forall\, k \in \mathbb{N}_0.$$

This completes the proof for the restricted form of Theorem 2.1. According to the value of m in the initial condition $y^{(m)}(0) = 0$, the structure of the total derivative (24) will change. Specifically, note that for $m = 0$ and $m = n - 1$, D_0 is given by

$$D_0 = \partial_x + y''\partial_{y'} + \cdots + \omega(x, \bar{F}, y', \ldots, y^{(n-1)})\partial_{y^{(n-1)}}$$

and

$$D_0 = \partial_x + y'\partial_y + \cdots + \hat{F}\partial_{y^{(n-2)}}$$

respectively, where $\bar{F}$ and $\hat{F}$ are the corresponding reduced equations. In every case, the identity

$$D_0 y^{(m-1)} \equiv y^{(m)}, \quad 1 \leq m \leq n,$$

is the crucial identity which allows us to show that D and D_0 are equivalent on solutions of the reduced equation and which can be differentiated with respect to the other various derivatives of y to find expressions for the $\omega_{y^{(k)}}$ to substitute into the LSC given by (4).

The generalisation to linear combinations of arbitrary-order initial conditions is obvious. As such, consider the following two cases:

1. The ODE (2) subject to

   $$y^{(n-1)}(0) + c_{n-1} y^{(n-2)}(0) + \cdots + c_2 y'(0) + c_1 y(0) = 0,$$

   where $c_i \in \mathbb{Q}$.

2. The ODE (2) subject to

   $$y^{(m)}(0) + c_m y^{(m-1)}(0) + \cdots + c_2 y'(0) + c_1 y(0) = 0,$$

   where $1 \leq m \leq n-2$ and $c_i \in \mathbb{Q}$.

The need to differentiate between these two cases remains the ability to write the total derivative D_0 for the associated reduced equations in the appropriate form, though the analysis remains very similar. For example, when $1 \leq m \leq n-2$, the reduced equation is

$$y^{(m)} + c_m y^{(m-1)} + \cdots + c_2 y' + c_1 y = F(x, y, \ldots, y^{(m-1)}, y^{(m+1)}, \ldots, y^{(n-1)}),$$

with $F_{y^{(n-1)}} \neq 0$. By definition, we know that the reduced equation is regular at the point $x = 0$. The total derivative is

$$D_0 = \partial_x + \cdots + \Big(F - c_m y^{(m-1)} - \cdots - c_2 y' - c_1 y\Big)\partial_{y^{(m-1)}} + \cdots + \omega|_{RE}\, \partial_{y^{(n-1)}}$$

where

$$\omega|_{RE} \equiv \omega\Big(x, y, \ldots, y^{(m-1)}, F - c_m y^{(m-1)} - \cdots - c_2 y' - c_1 y, y^{(m+1)}, \ldots, y^{(n-1)}\Big).$$

Hence the LSC is $\Gamma_0 = 0$, where

$$\Gamma_0 = D_0^m Q - \sum_{k=m+1}^{n-1} F_{y^{(k)}}(D_0^k Q) + \sum_{k=0}^{m-1} (c_{k+1} - F_{y^{(k)}})(D_0^k Q). \tag{31}$$

and the crucial identity is

$$D_0\Big(F - c_1 y^{(m-1)} - \cdots - c_{m-1} y' - c_m y\Big) \equiv y^{(m+1)}.$$

Using this formulation, it is straightforward to show that D_0 and D are equivalent on solutions of the reduced equation and find necessary expressions for the $\omega_{y^{(k)}}$ to substitute into (4). Since (31) is regular at $x = 0$, we can apply $\mathbf{H}$ and hence write down the determining system for the IVP in consideration. Performing similar analysis for the first case (leading-order initial condition $y^{(n-1)}$) exhausts the possible IVPs for which ω is a regular polynomial and thus completes the proof of Theorem 2.1. ■

In principle, Theorem 2.1 allows a user to find the symmetry generators of any IVP that falls under its remit. In practise, however, there are one or two apparent difficulties associated with the characteristics found through the theorem, highlighted in [9]. First, it appears that a user will have to solve an infinite number of PDEs in order to find the characteristics. In some cases, such as the first two examples if §3, the series clearly terminate but in others there may be no closed form. In such circumstances, this method is of no use since no technique currently exists that allows a user to construct the reduced equation for such characteristics. The examples of §2, however, underline how useful the method is: if one new symmetry is found, a user can construct the reduced equation and hence apply the usual symmetry analysis to the found equation and (perhaps) find a solution to the IVP.

One difficulty remains, exemplified by the structure of the symmetry generators

$$X_7 = \frac{y'}{x}\partial_y, \quad X_8 = \frac{yy'}{x}\partial_y, \tag{32}$$

in §2: how can it be that the LSC for a reduced equation can be written as a Taylor series about $x = 0$ whilst the characteristic - and hence generators such as (32) - cannot be obtained directly in such a fashion? We restrict attention to third-order ODEs subject to the initial condition $y'(0) = 0$ to explore an approach to this question. In such cases, we must find, at worse, the least power of x to expect in the characteristics of the reduced equation, for which we will use the method of Frobenius, and thus look at

$$DQ = \eta_x + (\eta_y - \xi_x)y' - y'^2\xi_y - \xi y''. \tag{33}$$

since $Q(x, y, y') = \eta(x, y) - y'\xi(x, y)$ and (33) expresses the invariance of $y'(0) = 0$ under the symmetries of the associated IVP.

Recall that the method of Frobenius provides a power series solution $y(x, \nu)$ to a given differential equation that contains a regular singularity assumed to be, without loss of generality, at the origin (for further details, see [12]). We write ξ and η as power series

$$\xi(x, y) = \sum_{k=0}^{\infty} \xi_k(y)x^{k+\nu} \quad \text{and} \quad \eta(x, y) = \sum_{k=0}^{\infty} \eta_k(y)x^{k+\nu}.$$

so the characteristic becomes

$$Q(x,y,y') = \sum_{k=0}^{\infty} \Big(\eta_k(y) - y'\xi_k(y)\Big)x^{k+\nu}, \tag{34}$$

for some value(s) of ν to be determined, and (33) becomes

$$\begin{aligned} DQ &= \sum_{k=0}^{\infty}\Big\{(k+\nu)\eta_k x^{k+\nu-1} + y'\left[\eta_k'(y)x^{k+\nu} - (k+\nu)\xi_k(y)x^{k+\nu-1}\right] \\ &\quad - y'^2\xi_k'(y)x^{k+\nu} - y''\xi_k(y)x^{k+\nu}\Big\}. \end{aligned} \tag{35}$$

To perform Frobenius analysis, we expand $y(x)$, $y'(x)$, $\xi_k(y)$, $\eta_k(y)$ etc. as power series about the point $x=0$. Using the chain rule to do so for $\xi_k(y)$ and $\eta_k(y)$, we substitute these expansions into (35) to give

$$\begin{aligned} DQ|_0 &= \sum_{k=0}^{\infty}(k+\nu)\left[\eta_k(\alpha) + \frac{x^2}{2!}\gamma\eta_k'(\alpha) + \cdots\right]x^{k+\nu-1} \\ &\quad - \sum_{k=0}^{\infty}(k+\nu)\left[x\gamma + \frac{x^2}{2!}\omega_0 + \cdots\right]\left[\xi_k(\alpha) + \frac{x^2}{2!}\gamma\xi_k'(\alpha) + \cdots\right]x^{k+\nu-1} \\ &\quad + \sum_{k=0}^{\infty}\left[x\gamma + \frac{x^2}{2!}\omega_0 + \cdots\right]\left[\eta_k'(\alpha) + \frac{x^2}{2!}\gamma\eta_k''(\alpha) + \cdots\right]x^{k+\nu} \\ &\quad - \sum_{k=0}^{\infty}\left[x\gamma + \frac{x^2}{2!}\omega_0 + \cdots\right]^2\left[\xi_k'(\alpha) + \frac{x^2}{2!}\gamma\xi_k''(\alpha) + \cdots\right]x^{k+\nu} \\ &\quad - \sum_{k=0}^{\infty}\left[\gamma + x\omega_0 + \cdots\right]\left[\xi_k(\alpha) + \frac{x^2}{2!}\gamma\xi_k'(\alpha) + \cdots\right]x^{k+\nu} \end{aligned} \tag{36}$$

where ω_0 is shorthand for $\omega(0,\alpha,0,\gamma)$ and α, γ are as in (7). As the index k increases, we can determine values of ν for which $\xi_k(y)$ and $\eta_k(y)$ are solutions of (36). The table below summarises the structure of the characteristics as determined by the value of ν:

if $\nu = -1$	then	$\eta_0(\alpha) = 0\ \forall\,\alpha$ $\xi_0(\alpha) \neq 0$ for some α $\eta_1(\alpha)$ either 0 or nonzero $\xi_1(\alpha) = \frac{1}{2}\eta_0'(\alpha)\ \forall\,\alpha$ $\eta_2(\alpha) = 0\ \forall\,\alpha$
if $\nu = 0$	then	$\eta_0(\alpha) \neq 0$ for some α $\xi_0(\alpha) = 0\ \forall\,\alpha$ $\eta_1(\alpha) = 0\ \forall\,\alpha$
if $\nu \in \big(0, 1\big]$	then	$\eta_0 = 0\ \forall\,\alpha$ $\eta_0(\alpha) \neq 0$ for some α
if $\nu > 1$	then	$DQ\big\|_0 = 0$ provides no constraint
if $\nu < -1$	then	$\eta_0(\alpha) = \xi_0(\alpha) = 0 \Rightarrow$ no solution

The first row of this table reveals that at worse we can expect $O(\frac{1}{x})$, corresponding to $\nu = -1$ and $k = 0$, in the characteristic, such that (34) has the following structure:

$$Q(x, y, y') = \sum_{k=0}^{\infty} \left(\eta_k(y) - \frac{y'}{x} \xi_k(y) \right) x^k. \tag{37}$$

Clearly, (37) admits the possibility of point-equivalent symmetry generators of the form (32) and the problem of finding the characteristics of the reduced equation is relatively straightforward. Since we can write

$$\frac{y'(x)}{x} = \sum_{r=0}^{\infty} w_r(\alpha, \gamma) x^r$$

such that $Q \equiv Q(x, y, \alpha, \gamma)$, (33) at the point $x = 0$ becomes

$$DQ|_0 = \eta_1(\alpha) - w_1(\alpha, \gamma)\xi_0(\alpha) - w_0(\alpha, \gamma)\xi_1(\alpha). \tag{38}$$

By looking at powers of γ in (38) we find a small system of differential equations that will allow us to determine η_1, ξ_0 and ξ_1. Similarly, the rest of the determining system splits into an over-determined system of PDEs for each k that allows us to find the remaining η_k and ξ_k. It is this approach that is utilised in the examples of §3.

5 Further work

The method presented in this paper extends **H** to allow a user to solve any given regular IVP, although the practical application of the determining systems may not be straightforward. Clearly, we would like to be able to extend this method to include all possible initial conditions, most notably for some governing equation subject to

$$y(0) = 0.$$

Furthermore, the reliance of the analysis on Taylor series means that it is only valid for *regular* ODEs (i.e. ODEs that are regular in x and the argument of the initial condition). If it were possible to amend the method to incorporate ODEs with regular singularities then the scope of the method would be considerably more appealing.

Concerning linear combinations of arbitrary-order initial conditions, it would be useful if a user could know *a priori* if a particular combination of initial conditions is guaranteed to produce at least one new point-equivalent symmetry generator. Recently, "consistency" conditions have been introduced [1, 7] that allow a user to determine what boundary conditions are compatible with certain properties of a given differential equation. In a similar fashion, to be able to choose the constants c_i such that the associated linear combination of initial conditions admitted new symmetries would be particularly useful.

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