Determining the point-equivalent symmetries of initial-value problems

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Abstract

Although symmetry analysis provides a powerful tool for solving differential equations, it has not proved to be so successful in the treatment of initial- and boundary-value problems. A possible reason for this is the belief that the set of symmetries of an initial-value problem is a subset of the symmetries of the governing differential equation. It was recently shown that this is, in fact, not the case and a method was introduced for constructing the symmetries of a class of initial-value problems using Taylor series. We extend this method for arbitrary-order regular ordinary differential equations subject to both an arbitrary-order single initial condition and an arbitrary linear combination of initial conditions. Furthermore, we propose a practical method for dealing with a class of IVPs that possess a regular singularity through the use of the Frobenius method.

1 Introduction

Towards the end of the 19th century, the Norwegian mathematician Sophus Lie developed an astonishing theory that provided a fruitful, systematic mechanism for solving differential equations. Although Lie's approach and its many extensions have had great success in determining and classifying solutions of differential equations, it has proved much less successful in the treatment of boundary- and initial-value problems (BVPs and IVPs respectively). Indeed, the existence and determination of a "general procedure for applying symmetry methods to BVPs" is considered one of the significant open problems in the area of symmetry analysis (ref).

The prevailing view suggests that symmetry groups admitted by B- and IVPs are not sufficiently rich to allow for the effective use of symmetry reduction. This view is informed by the belief that a one-parameter Lie group of transformations is admitted by a a BVP if and only if it leaves invariant the

1. boundary
2. boundary conditions
3. governing differential equation.

Which is to assume that the group of symmetries of a BVP are a subset of the group of symmetries of the governing equation from which the BVP is formed. Using this assumption as a base, current research methods have concentrated on specific approaches that do not employ a familiar, reductive algorithm typical of Lie's method. Instead, these methods largely concentrate on the alternative problem of classifying those initial conditions that, through the application of symmetry methods, will solve classes of associated IVPs.

For example, some IVPs can be solved through the use of an iterative approach (goard). Starting with a given IVP and armed with knowledge of (possibly trivial) solutions of the governing PDE, the method uses the infinite-dimensional part of the Lie algebra to build a hierarchy of associated IVPs until one is found that can be solved. Once armed with this solution, a user can work back along the chain to find the symmetries of the original IVP.

In an attempt to negate the prevailing view that classical symmetries are insufficiently rich to solve B- and IVPs, much effort has been concentrated on higher-order and nonclassical symmetry techniques (zhdanov). Such techniques show that higher-order conditional symmetries are responsible for the reduction of IVPs to Cauchy problems for some system of differential equations, under appropriate ansatze concerning the structure of solutions of the invariant surface condition. Thus, a user can classify those IVPs that can be reduced to Cauchy problems, a classification that is performed for evolution equations in two independent variables.

Although much research effort has been expended concerning B- and IVPs, there remains no general, systematic procedure a user can employ to find symmetries that satisfy conditions (1)-(3). Instead of this being the fault of the various approaches, perhaps the difficulty lies in the conditions themselves? Clearly, for an IVP, the boundary data must be invariant under any symmetry and the domain of the problem must be mapped to itself; if this were not the case the problem would be mapped to a different B- or IVP. Condition (3) is not necessary. If a given differential equation admits a one-parameter Lie group of transformations then it can be reduced in order by one, plus one quadrature; similarly, if a given differential equation is prescribed an initial condition then, once the general solution is known, the IVP will leave a differential equation of reduced order - one less than the order of the governing differential equation. In effect, the initial condition has reduced the order of the differential equation. However, the symmetries of the reduced equation do not have to coincide with the symmetries of the governing equation - indeed, they may have no symmetries in common at all. Any symmetries of the the reduced equation, however, will necessarily be symmetries of the IVP generated by the governing equation subject to the initial condition and will thus leave the initial condition invariant.

Equivalently, denote the set of all solutions of the governing equation by S; let $T \subset S$ denote the set of all solutions of the governing equation that also satisfy the initial conditions. Every symmetry maps S to itself; however, we are not interested in all solutions of the governing equation but only those that satisfy the initial conditions and thus the IVP. Hence, we require only that any symmetries of the IVP map the subset T to itself - any solutions that fall outside this domain, namely those in $S\backslash T$, do not need to be mapped to other solutions.

Using this observation as a basis, Hydon (reference) constructs a systematic method by which it is possible to construct an equation that is satisfied by all solutions of T but not by all solutions of S, and hence find the symmetries of a given IVP. So far, this method has proved successful for a particular class of IVPs, namely

$$y''' = \omega(x, y, y', y'') \quad \text{subject to} \quad y''(0) = 0.$$

In this paper, we extend the method of Hydon to incorporate several other classes of IVPs for regular polynomial ordinary differential equations.

In §2, we employ a simple example to demonstrate why the condition (3) is not required. In §3, we adjust the method of Hydon to construct the symmetries of a class of IVPs and use this result to generalise the method for arbitrary-order ODEs complete with arbitrary-order linear combinations of initial conditions. Furthermore, we provide analysis that will allow the solution of a particular class of different equations in which regular singularities appear through the use of the method of Frobenius. In §4 we provide examples which illustrate the method in action and suggest, in §5, further extensions of the work and open problems.

2 An example

We consider the initial-value problem (henceforth IVP) given by

$$y''' = \omega(x, y, y', y'') \quad \text{subject to} \quad y'(0) = 0. \tag{1}$$

Recall that dynamical symmetries of a given nth-order ordinary differential equation (ODE) are generated by X, where

$$X = Q(x, y, y')\partial_y \quad \text{and} \quad Q = \eta(x, y) - y'\xi(x, y).$$

We restrict attention here to third-order ODEs and those dynamical symmetries that are equivalent to point-symmetries such that the characteristic $Q(x, y, y')$ is linear in y'. Such symmetries will be referred to as point-equivalent symmetries. Recall also that the total derivative, D, with respect to x, is defined on solutions of $y''' = \omega(x, y, y', y'')$ as

$$D = \partial_x + y'\partial_y + y''\partial_{y'} + \omega(x, y, y', y'')\partial_{y''}. \qquad (2)$$

where $\partial_x = \partial/\partial x$ etc.

The method for obtaining the dynamical symmetries of a given third-order ODE is well known (references). Using the second prolongation of the point-equivalent symmetry generator X,

$$X^{(2)} = Q\partial_y + DQ\partial_{y'} + D^2Q\partial_{y''}, \qquad (3)$$

the characteristics Q are found by solving the linearised symmetry condition

$$D^3Q - X^{(2)}\Big(\omega(x, y, y', y'')\Big) = 0.$$

The solution of the linearised symmetry condition can be found since it splits into an overdetermined system of partial differential equations for $\xi(x, y)$ and $\eta(x, y)$ by virtue of the fact these functions are independent of derivatives of y. Thus, in the case of the simplest third-order ODE, namely

$$y''' = 0, \qquad (4)$$

it can be shown that the Lie algebra of point-equivalent symmetry generators has a 7-dimensional basis, spanned by

$$X_1 = \partial_y, \quad X_2 = y\partial_y, \quad X_3 = x^2\partial_y, \quad X_4 = -xy'\partial_y,$$
$$X_A = -y'\partial_y, \quad X_B = x\partial_y, \quad X_C = (2xy - x^2y')\partial_y.$$

Note that the first integrals of the *governing* equation (4) are given by

$$\alpha = y - xy' + \frac{1}{2}x^2y'', \quad \beta = y' - xy'', \quad \gamma = y''. \tag{5}$$

(Recall that a non-constant function $\alpha(x, y, y', y'')$ is a first integral of an ODE if and only if it is constant along solutions of the ODE i.e. if and only if $D\alpha = 0$).

When $\omega = 0$, the IVP (1) is that ODE for which $\beta = 0$, which is to say

$$y'' = \frac{y'}{x}. \tag{6}$$

This *reduced* equation has an *eight*-dimensional Lie algebra of point-equivalent symmetry generators, spanned by

$$X_1 = \partial_y, \quad X_2 = y\partial_y, \quad X_3 = x^2\partial_y, \quad X_4 = -xy'\partial_y,$$
$$X_5 = (2y^2 - xyy')\partial_y, \quad X_6 = (2x^2y - x^3y')\partial_y, \quad X_7 = \tfrac{y'}{x}\partial_y, \quad X_8 = \tfrac{yy'}{x}\partial_y.$$

The symmetry generators $X_1 - X_4$ coincide with the symmetries of (4) whilst $X_5 - X_8$ are new symmetries; note also that X_A, X_B and X_C are not symmetries of (6). By working in terms of the first integrals (5), it is possible to show why X_A, X_B and X_C are not symmetries of the reduced equation and why the new generators $X_5 - X_8$ are.

The reduced equation of the IVP (1) (with $\omega = 0$) is given by (6) and its first integrals by

$$\alpha_0 = \alpha|_{\beta=0} = y - \frac{1}{2}xy', \quad \gamma_0 = \gamma|_{\beta=0} = \frac{y'}{x}.$$

In the space of first integrals, dynamical symmetry generators act as point transformations [p.129, Hydon]; hence, when the symmetry generators are written in terms of first integrals, they are independent of x. By use of the chain-rule, a generator X of dynamical symmetries can be written in terms of the first integrals as follows:

$$X = (X^{(2)}\alpha)\partial_\alpha + (X^{(2)}\beta)\partial_\beta + (X^{(2)}\gamma)\partial_\gamma$$

where $X^{(2)}$ is given by (3). Hence, in terms of its first integrals, the ODE (4) has point-equivalent symmetry generators given by

$$X_1 = \partial_\alpha, \quad X_2 = \alpha\partial_\alpha + \beta\partial_\beta + \gamma\partial_\gamma, \quad X_3 = 2\partial_\gamma, \quad X_4 = \beta\partial_\beta + 2\gamma\partial_\gamma,$$
$$X_A = \beta\partial_\alpha + \gamma\partial_\beta, \quad X_B = \partial_\beta, \quad X_C = 2\alpha\partial_\beta + 2\beta\partial_\gamma.$$

Careful consideration of these symmetry generators reveals why $X_A - X_C$ do not generate symmetries of the reduced equation $\beta = 0$: whilst $X_1 - X_4$ leave the submanifold $\beta = 0$ invariant, $X_A - X_C$ do not; for example, X_B generates translations in the the β-direction such that the plane $\beta = 0$ is mapped to $\beta = c$, where c is a non-zero constant. Similarly, X_A generates translations in the β direction that depend upon the value of γ.

The new symmetries of the reduced ODE (6), written in terms of the first integrals, are

$$X_5 = (2\alpha^2 + \tfrac{1}{2}x^3\beta\gamma)\partial_\alpha + 3\beta(-\tfrac{1}{2}x^2\gamma + \alpha)\partial_\beta + (2\beta^2 + 3x\beta\gamma + 2\alpha\gamma)\partial_\gamma,$$
$$X_6 = x^3\beta\partial_\alpha - 3x^2\beta\partial_\beta + 2(2\alpha + 3x\beta)\partial_\gamma,$$
$$X_7 = (\tfrac{3\beta}{x} + \gamma)\partial_\alpha - 3\tfrac{\beta}{x^2}\partial_\beta + 2\tfrac{\beta}{x^3}\partial_\gamma,$$
$$X_8 = (3\tfrac{\alpha\beta}{x} + \alpha\gamma + \beta^2)\partial_\alpha + 3\alpha\beta(-\tfrac{1}{x^2} + \tfrac{1}{2})\partial_\beta + (2\tfrac{\alpha\beta}{x^3} + \gamma^2)\partial_\gamma.$$

Since they clearly depend on x, these are not symmetries of the governing equation (4); on the submanifold $\beta = 0$, however, the symmetries become

$$X_5 = 2\alpha_0^2\partial_{\alpha_0} + 2\alpha_0\gamma_0\partial_{\gamma_0},$$
$$X_6 = 4\alpha_0\partial_{\gamma_0},$$
$$X_7 = \gamma_0\partial_{\alpha_0},$$
$$X_8 = \alpha_0\gamma_0\partial_{\alpha_0} + \gamma_0^2\partial_{\gamma_0}.$$

Not only are these independent of x but they also leave the submanifold $\beta = 0$ invariant and hence are generators of dynamical symmetries of the reduced ODE.

This example clearly demonstrates that the set of all symmetries of a reduced equation is *not* a subset of the symmetries of the associated governing equation. Indeed, they may have no symmetries in common. We require only those solutions of the governing equation that satisfy the imposed initial condition; such solutions (and the initial condition itself) will be left invariant by the symmetries of the IVP.

Until recently, no technique existed which allowed a user to find constructively the symmetries of an IVP. Hydon, however, has recently proposed such a technique that enables the user to systematically find the point-equivalent symmetries of a class of IVPs, defined by

$$y''' = \omega(x, y, y', y'') \quad \text{subject to} \quad y''(0) = 0,$$

utilising a Taylor series expansion about the point $x = 0$. This technique, however, does not allow a user to find symmetry generators of the form exmeplified by X_7 and X_8 of the reduced equation (6), namely

$$X_7 = \frac{y'}{x}\partial_y \quad \text{and} \quad X_8 = \frac{yy'}{x}\partial_y. \tag{7}$$

This is because they are the result of a regular singularity in the reduced equation and hence cannot be found directly through the use of Taylor series. We propose a modification of the technique such that a user can find the point-equivalent symmetry generators of the class of IVPs given by (1).

3 Symmetry construction of a class of initial-value problems

We consider the class of initial-value problems defined by

$$y''' = \omega(x, y, y', y'') \tag{8}$$

subject to the initial condition

$$y'(0) = 0. \tag{9}$$

The total derivative with respect to x on solutions of the governing equation is given by (2) and the linearised symmetry condition (henceforth LSC) for (8) is $\Gamma = 0$, where

$$\Gamma = D^3Q - \omega_{y''}(D^2Q) - \omega_{y'}(DQ) - \omega_y Q. \tag{10}$$

The first integrals of (8) are chosen to satisfy (without loss of generality)

$$\alpha(x, y, y', y'') = y(0), \quad \beta(x, y, y', y'') = y'(0), \quad \gamma(x, y, y', y'') = y''(0).$$

As shown in the example, the reduced equation is therefore given by $\beta = 0$ and can be written in the form

$$y'' = \tilde{F}(x, y, y') \tag{11}$$

Note specifically that the reduced equation $y'' = \tilde{F}(x, y, y')$ given by (6) has a regular singularity at $x = 0$. On solutions of the reduced equation (11), the total derivative D is replaced by

$$\tilde{D}_0 = \partial_x + y'\partial_y + \tilde{F}(x, y, y')\partial_{y'}$$

and the LSC for (11) becomes $\tilde{\Gamma}_0 = 0$, where

$$\tilde{\Gamma}_0 = \tilde{D}_0^2 Q - \tilde{F}_{y'}(\tilde{D}_0 Q) - \tilde{F}_y Q.$$

Hydon showed that, for the IVP $y''' = \omega(x, y, y', y'')$ subject to $y''(0) = 0$, the LSC for the associated reduced equation $y'' = G(x, y, y')$ could be written as a Taylor series about $x = 0$. As a direct result of this, it was possible to find a system of equations - known as the *determining system* - that would allow the explicit calculation of the point-equivalent symmetry generators of the reduced equation. In this case, however, the reduced equation (11) has a regular singularity, meaning it is not possible to write its LSC, $\tilde{\Gamma} = 0$, as a Taylor series and thus we cannot apply the new method in its current form.

Let us consider instead the reduced equation in its "unsolved" form

$$y' = F(x, y, y''). \tag{12}$$

Now F no longer has a regular singularity at the point $x = 0$ since, by definition,

$$y'(0) = F(0, \alpha, \gamma) \equiv 0 \quad \forall\, \alpha, \gamma.$$

The total derivative D on solutions of (12) is replaced by

$$D_0 = \partial_x + F(x, y, y'')\partial_y + \omega(x, y, F(x, y, y''), y'')\partial_{y''}$$

and, since $Q = Q(x, y, y')$, we define

$$\begin{aligned} Q|_{y'=F(x,y,y'')} &= \eta(x, y) - F(x, y, y'')\xi(x, y) \\ &= Q_0. \end{aligned}$$

Hence, the LSC becomes $\Gamma_0 = 0$, where

$$\Gamma_0 = D_0 Q_0 - F_{y''}(D_0^2 Q_0) - F_y Q_0. \tag{13}$$

We can thus write (13) as a Taylor series about the point $x = 0$ as required by the method of Hydon and hence show it is possible to solve (13) even though the reduced equation (12) is unknown.

Given an arbitrary differentiable function $H(x, y, y', y'')$, define

$$h(x, y, y'') = H(x, y, F(x, y, y''), y'').$$

Using the crucial identity

$$D_0 F \equiv y'' \tag{14}$$

and the chain rule, it is straightforward to show that

$$D_0 h(x, y, y'') = \left(DH(x, y, y', y'')\right)\big|_{y'=F(x,y,y'')}$$

and, by induction,

$$D_0^k h(x, y, y'') = \left(D^k H(x, y, y', y'')\right)\big|_{y'=F(x,y,y'')}, \quad \forall k \in \mathbb{N} \tag{15}$$

Which is to say D and D_0 are equivalent when D is projected onto the submanifold defined by (12). Using these relationships, we have the following theorem:

Theorem 3.1. *The projection of the linearised symmetry condition of the governing (third-order) ODE onto solutions of the (second-order)reduced equation provides the following relationships:*

$$\Gamma\big|_{y'=F(x,y,y'')} = -\frac{1}{F_{y''}} D_0\Gamma_0 - \left(\frac{F_y}{F_{y''}} + (\omega_{y'})\big|_{y'=F(x,y,y'')}\right)\Gamma_0 \tag{16}$$

and

$$(D^k\Gamma)\big|_{y'=F(x,y,y'')} = D_0^k\left\{-\frac{1}{F_{y''}} D_0\Gamma_0 - \left(\frac{F_y}{F_{y''}} + (\omega_{y'})\big|_{y'=F(x,y,y'')}\right)\Gamma_0\right\}$$

for each $k \in \mathbb{N}$.

Proof. To prove this result, note the identity (14) and differentiate it with respect to y and y'' respectively in order to find expressions for ω_y and $\omega_{y''}$ to substitute into (10). To complete the proof, apply (15) to (16). ■

Since the reduced equation (12) is regular in x, we can write its LSC as a Taylor series about $x = 0$ as follows:

$$\Gamma_0 = \sum_{k=0}^{\infty} \frac{x^k}{k!}\left(D_0^k\Gamma_0\right)\Big|_0 = 0 \qquad \forall\, k \in \mathbb{N}_0.$$

where $|_0$ represents replacing (x, y, y', y'') by $(0, \alpha_0, 0, \gamma_0)$ respectively. Note that it would not be possible to perform this important step if we continued using the reduced equation in the form (11).

When $k = 0$, we have

$$\Gamma_0\Big|_0 = 0. \tag{17}$$

Recall that the LSC for the reduced equation is given by (12). About the point $x = 0$, noting that $F(0, \alpha_0, \gamma_0) \equiv 0$ and $Q\big|_0 \equiv Q_0$, and recalling the induction result(15), we see that (17) becomes

$$\Gamma_0\Big|_0 = (D_0Q_0)\Big|_0 = (DQ)\Big|_0 = 0.$$

For all other $k \geq 0$, Theorem 3.1 allows us to conclude that

$$(D^k\Gamma)\Big|_0 = 0 \quad \forall\, k \in \mathbb{N}_0.$$

We have thus shown that it is possible to write down the LSC for the reduced equation *in terms of things we do know*, despite not knowing the reduced equation itself and hence the total derivative on solutions of the reduced equation. The following theorem provides the conditions necessary to solve, in principle, the LSC of the IVP given by (8) and (9).

Theorem 3.2. *The point-equivalent symmetry generators $X = Q\partial_y$ for the IVP $y''' = \omega(x, y, y', y'')$ subject to $y'(0) = 0$ are obtained by solving the determining system*

$$\big(DQ\big)\big|_0 = 0 \quad \textit{and} \quad \big(D^k\Gamma\big)\big|_0 = 0 \quad \forall\ k \in \mathbb{N}_0, \tag{18}$$

where Γ is given by (10)*.*

Note that the first condition expresses the invariance of the initial condition under the symmetry generators of the IVP and constrains the remaining equations of the determining system.

There are one or two apparent difficulties associated with the characteristics found by the determining system in Theorem 3.2. First, the system is for all $k \in \mathbb{N}_0$, so it appears there will be an infinite series of partial differential equations to solve in order to find the characteristics. This is not necessarily a concern, for in some cases the series obviously terminate, leaving the user with finite equations with which to deal. Second, on those occasions when the series do not obviously terminate and is either difficult to recognise or does not have a finite form, note that we require only *one* new symmetry generator - that isn't a symmetry generator of the governing equation - in order for the method to be useful. In such an instance, this symmetry generator delivers symmetries of the reduced equation (12), from which we can use the theory of differential invariants to construct the reduced equation itself. Once we have the reduced equation, we can perform the standard symmetry analysis to find the remaining symmetries. The usefullness of this method is therefore underlined: should it yield just one new characteristic in closed form the user will be able to reconstruct the reduced equation of the IVP and thus find all the remaining symmetry generators and (perhaps) a solution to the IVP.

The determining system in Theorem 3.2 clearly needs to produce generators that include those of the form (7). This presents us with something of a problem since such a method follows somewhat illogically from the analysis presented so far: how can it be that the linearised symmetry condition for the reduced equation can be written as a Taylor series about $x = 0$ whilst the characteristic $Q(x, y, y')$ - and hence generators such as X_7 and X_8 - cannot be obtained directly in such a fashion?

In order to determine the characterstic $Q(x, y, y')$ in such cases, we first need to answer the question of what to expect from the IVP in consideration: at worse, what is the least power of x can we expect in the characteristics of the reduced equation? The method of Frobenius provides us with an answer.

In what follows, we will adopt the notation of Hydon throughout, namely,

$$\eta_k(\alpha) = \left.\frac{\partial^k \eta(x,\alpha)}{\partial x^k}\right|_{x=0}, \quad \xi_k(\alpha) = \left.\frac{\partial^k \xi(x,\alpha)}{\partial x^k}\right|_{x=0}, \quad k \in \mathbb{N}_0,$$

with derivatives of $\eta_k(\alpha)$ and $\xi_k(\alpha)$ with respect to α_0 denoted by η_k', η_k'', ξ_k' etc. Furthermore, the argument α will, in general, be left out; we also assume that $y(x)$ is analytic about the point $x = 0$ so that we can write it as a Taylor series. Since $Q(x, y, y') = \eta(x, y) - y'\xi(x, y)$, we have

$$DQ = \eta_x + (\eta_y - \xi_x)y' - y'^2\xi_y - \xi y''. \tag{19}$$

Recall that the method of Frobenius provides a power series solution $y(x, \nu)$ to a given differential equation that contains a regular singularity assumed to be, without loss of generality, at the origin (for further details, see Ince).

We write η and ξ as power series

$$\eta(x, y) = \sum_{k=0}^{\infty} \eta_k(y)x^{k+\nu} \quad \text{and} \quad \xi(x, y) = \sum_{k=0}^{\infty} \xi_k(y)x^{k+\nu}.$$

so the characteristic becomes

$$Q(x,y,y') = \sum_{k=0}^{\infty} \Big(\eta_k(y) - y'\xi_k(y) \Big) x^{k+\nu},$$

for some value(s) of ν to be determined, and (19) becomes

$$\begin{aligned} DQ &= \sum_{k=0}^{\infty} \Big\{ (k+\nu)\eta_k x^{k+\nu-1} + y' \left[\eta_k'(y) x^{k+\nu} - (k+\nu)\xi_k(y) x^{k+\nu-1} \right] \\ &\quad - y'^2 \xi_k'(y) x^{k+\nu} - y''\xi_k(y) x^{k+\nu} \Big\}. \end{aligned} \tag{20}$$

To perform Frobenius analysis, we expand $y(x)$, $y'(x)$, $\xi_k(y)$, $\eta_k(y)$ etc. as power series about the point $x=0$. Using the chain rule to do so for $\xi_k(y)$ and $\eta_k(y)$, we substitute these expansions into (20) to give

$$\begin{aligned} DQ|_0 &= \sum_{k=0}^{\infty} (k+\nu) \left[\eta_k(\alpha) + \frac{x^2}{2!}\gamma\eta_k'(\alpha) + \cdots \right] x^{k+\nu-1} \\ &\quad - \sum_{k=0}^{\infty} (k+\nu) \left[x\gamma + \frac{x^2}{2!}\omega_0 + \cdots \right] \left[\xi_k(\alpha) + \frac{x^2}{2!}\gamma\xi_k'(\alpha) + \cdots \right] x^{k+\nu-1} \\ &\quad + \sum_{k=0}^{\infty} \left[x\gamma + \frac{x^2}{2!}\omega_0 + \cdots \right] \left[\eta_k'(\alpha) + \frac{x^2}{2!}\gamma\eta_k''(\alpha) + \cdots \right] x^{k+\nu} \\ &\quad - \sum_{k=0}^{\infty} \left[x\gamma + \frac{x^2}{2!}\omega_0 + \cdots \right]^2 \left[\xi_k'(\alpha) + \frac{x^2}{2!}\gamma\xi_k''(\alpha) + \cdots \right] x^{k+\nu} \\ &\quad - \sum_{k=0}^{\infty} \left[\gamma + x\omega_0 + \cdots \right] \left[\xi_k(\alpha) + \frac{x^2}{2!}\gamma\xi_k'(\alpha) + \cdots \right] x^{k+\nu} \end{aligned} \tag{21}$$

where ω_0 is shorthand for $\omega(0,\alpha,0,\gamma)$. As the index k increases, we can determine values of ν for which $\xi_k(y)$ and $\eta_k(y)$ are solutions of (21). The table below summarises the structure of the characteristics as determined by the value of ν:

if $\nu = -1$	then	$\eta_0(\alpha) = 0 \;\forall\, \alpha$ $\xi_0(\alpha) \neq 0$ for some α $\eta_1(\alpha)$ either 0 or nonzero $\xi_1(\alpha) = \frac{1}{2}\eta_0'(\alpha) \;\forall\, \alpha$ $\eta_2(\alpha) = 0 \;\forall\, \alpha$
if $\nu = 0$	then	$\eta_0(\alpha) \neq 0$ for some α $\xi_0(\alpha) = 0 \;\forall\, \alpha$ $\eta_1(\alpha) = 0 \;\forall\, \alpha$
if $\nu \in \big(0, 1\big]$	then	$\eta_0 = 0 \;\forall\, \alpha$ $\eta_0(\alpha) \neq 0$ for some α
if $\nu > 1$	then	$DQ\big\|_0 = 0$ provides no constraint
if $\nu < -1$	then	$\eta_0(\alpha) = \xi_0(\alpha) = 0 \Rightarrow$ no solution

The first row of this table reveals that at worse we can expect $O(\frac{1}{x})$, corresponding to $\nu = -1$ and $k = 0$, in the characteristic, such that $Q(x, y, y')$ has the following structure:

$$Q(x, y, y') = \sum_{k=0}^{\infty} \left(\eta_k(y) - \frac{y'}{x}\xi_k(y) \right) x^k. \tag{22}$$

Clearly, (22) admits the possibility of point-equivalent symmetry generators of the form (7) and the problem of finding the characteristics of the reduced equation is relatively straightforward.

Since we can write

$$\frac{y'(x)}{x} = \sum_{r=0}^{\infty} w_r(\alpha, \gamma) x^r$$

such that $Q \equiv Q(x, y, \alpha, \gamma)$, (19) at the point $x = 0$ becomes

$$DQ\big|_0 = \eta_1(\alpha) - w_1(\alpha, \gamma)\xi_0(\alpha) - w_0(\alpha, \gamma)\xi_1(\alpha). \tag{23}$$

By looking at powers of γ in (23) we find a small system of differential equations that will allow us to determine η_1, ξ_0 and ξ_1. Similarly, the system

$$D^k \Gamma\big|_0 = 0 \quad \forall\, k \in \mathbb{N}_0.$$

splits into an over-determined system of PDEs for each k that allows us to find the remaining η_k and ξ_k. In such a way we can find, in practice, the point-equivalent symmetry generators of the appropriate form using the determining systems we have systematically found in our analysis.

Examples of the process outlined above can be found in §4. Before that, however, we can generalise the analysis presented so far. Suppose we have the IVP

$$y^{(n)} = \omega(x, y, y', \dots, y^{(n-1)}) \tag{24}$$

subject to the initial condition

$$y^{(m)}(0) = 0, \quad 0 \le m < n. \tag{25}$$

The LSC for the unconstrained problem (24) remains $\Gamma = 0$, where

$$\begin{aligned} \Gamma &= D^n Q - \omega_{y^{(n-1)}}(D^{n-1}Q) - \cdots - \omega_{y'}(DQ) - \omega_y Q \\ &= D^n Q - \sum_{k=0}^{n-1} \omega_{y^{(k)}}(D^k Q). \end{aligned} \tag{26}$$

Clearly, the structure of the reduced equation and the total derivative, D_0, on solutions of the reduced equation depend on the vaule of m in (25). For example, if $m = 0$, then we write the reduced equation as

$$y = F\big(x, y', y'', \dots, y^{(n-1)}\big), \quad F_{y^{(n-1)}} \ne 0,$$

with total derivative

$$D_0 = \partial_x + y''\partial_{y'} + \cdots + \omega(x, F, y', \dots, y^{(n-1)})\partial_{y^{(n-1)}}$$

and the LSC $\Gamma_0 = 0$, where

$$\begin{aligned} \Gamma_0 &= Q - F_{y^{(n-1)}}(D_0^{n-1}Q) - F_{y^{(n-2)}}(D_0^{n-2}Q) - \cdots - F_{y'}(D_0 Q) \\ &= Q - \sum_{k=1}^{n-1} F_{y^{(k)}}(D_0^k Q). \end{aligned}$$

The only other cases to consider are

1. $1 \leq m \leq n-2$
2. $m = n-1$.

Such alterations, however, do not fundamentally alter the process for finding the determining system of the point-equivalent symmetries of the IVP in general. By noting the identity

$$D_0 y^{(m-1)} \equiv y^{(m)}, \qquad \text{for } 1 \leq m \leq n \tag{27}$$

it is easy to show that D and D_0 are equivalent on solutions of the reduced equation. Furthermore, by differentiating (27) with respect to the various other derivatives of y, we can find expressions for the $\omega_{y^{(k)}}$ to substitute into the LSC given by (26). Thus, using the assumption that we can write the LSC as a Taylor series about the point $x = 0$, it is therefore possible to find an analogy to Theorem 3.2 and write down the determining system for the problem.

We can further generalise the process. IVPs are often posed with a complete set of initial conditions and, as such, it is worth considering whether linear combinations of initial conditions could possibly yield new point-equivalent symmetries to determine solutions of such IVPs. To find the determining systems of the point-equivalent symmetries of such systems, we follow the by now familiar path.

Consider the arbitrary-order ODE given by (24). In order to generalise the analysis it is necessary to consider two seperate cases:

1. (24) subject to

$$y^{(n-1)}(0) + c_1 y^{(n-2)}(0) + \cdots + c_{n-2} y'(0) + c_{n-1} y(0) = 0,$$

   where $c_i \in \mathbb{Q}$ and $i = 1 \ldots n-1$

2. (24) subject to

$$y^{(m)}(0) + c_1 y^{(m-1)}(0) + \cdots + c_{m-1} y'(0) + c_m y(0) = 0,$$

   where $1 \leq m \leq n-2$, $c_i \in \mathbb{Q}$ and $i = 1 \ldots m$.

The need to differentiate between the two cases remains the ability to write the total derivative for the reduced equation in the appropriate form. However, the analysis remains very similar. For example, in the case where the leading-order derivative of the initial condition is $1 \leq m \leq n-2$, the reduced equation is given by

$$y^{(m)} + c_1 y^{(m-1)} + \cdots + c_{m-1} y' + c_m y = F(x, y, \ldots, y^{(m-1)}, y^{(m+1)}, \ldots, y^{(n-1)}),$$

with $F_{y^{(n-1)}} \neq 0$ and so, by definition, we know that the reduced equation is regular at the point $x = 0$. The total derivative is

$$D_0 = \partial_x + \cdots + \Big(F - c_1 y^{(m-1)} - \cdots - c_{m-1} y' - c_m y\Big)\partial_{y^{(m-1)}} + \cdots + \omega|_{RE}\,\partial_{y^{(n-1)}}$$

where

$$\omega|_{RE} \equiv \omega\Big(x, y, \ldots, y^{(m-1)}, F - c_1 y^{(m-1)} - \cdots - c_{m-1} y' - c_m y, y^{(m+1)}, \ldots, ^{(n-1)}\Big).$$

Hence the LSC is $\Gamma_0 = 0$, where

$$\Gamma_0 = D_0^m Q - \sum_{k=m+1}^{n-1} F_{y^{(k)}}(D_0^k Q) + \sum_{k=0}^{m-1}(c_{m-k} - F_{y^{(k)}})(D_0^k Q).$$

The crucial identity we require is

$$D_0\Big(F - c_1 y^{(m-1)} - \cdots - c_{m-1} y' - c_m y\Big) \equiv y^{(m+1)}.$$

Once again, using this identity we can show that D_0 and D are equivalent on solutions of the reduced equation and, furthermore, by differentiating it with respect to the various other derivatives of y, find expressions for the ω_{y^k} to substitute into (26).

Once the various steps outlined above are completed, one exhausts the possible initial-value problems in which ω is a regular polynomial. The following three theorems provide the determining systems for the point-equivalent symmetries of the corresponding IVP:

Theorem 3.3. *For the initial-value problem given by* (24) *subject to the linear combination of initial conditions*

$$y^{(n-1)}(0) + c_1 y^{(n-2)}(0) + \cdots + c_{n-2} y'(0) + c_{n-1} y(0) = 0,$$

for $c_i \in \mathbb{Q}$, $i = 1 \ldots n-1$, *the restriction of the linearised symmetry condition of the governing equation to solutions of the reduced equation provides the following relationships:*

$$\Gamma\big|_{RE} = D_0\Gamma_0 + \Big(F_{y^{(n-2)}} - \omega_{y^{(n-1)}}\big|_{RE} - c_1\Big)\Gamma_0$$

and

$$D^k\Gamma\Big|_{RE} = D_0^k\Big\{D_0\Gamma_0 + \big(F_{y^{(n-2)}} - \omega_{y^{(n-1)}}\big|_{RE} - c_1\big)\Gamma_0\Big\}, \quad k \in \mathbb{N}_0,$$

where RE *represents the reduced equation and* Γ, Γ_0 *are the linearised symmetry conditions of the governing and reduced equations respectively. Furthermore, the point-equivalent symmetry generators* $X = Q\partial_y$ *for the IVP are obtained by solving the system*

$$\Big(D^{n-1}Q + c_1(D^{n-2}Q) + \cdots + c_{n-2}(DQ) + c_{n-1}Q\Big)\Big|_0 = 0$$

and

$$D^k\Gamma\Big|_0 = 0 \quad \forall k \in \mathbb{N}_0.$$

Theorem 3.4. *For the initial-value problem given by* (24) *subject to the linear combination of initial conditions*

$$y^{(m)}(0) + c_1 y^{(m-1)}(0) + \cdots + c_{m-1} y'(0) + c_m y(0) = 0, \quad 1 \leq m \leq n-2$$

for $c_i \in \mathbb{Q}$, $i = 1 \ldots n-1$, *the restriction of the linearised symmetry condition of the governing equation to solutions of the reduced equation provides the following relationships:*

$$\Gamma\big|_{RE} = \left(\frac{1}{F_{y^{(n-1)}}}(c_1 - F_{y^{(m-1)}}) - \omega_{y^{(m)}}\big|_{RE}\right)\Gamma_0 - \frac{1}{F_{y^{(n-1)}}}D_0\Gamma_0$$

and

$$D^k\Gamma\Big|_{RE} = D_0^k\left\{\left(\frac{1}{F_{y^{(n-1)}}}(c_1 - F_{y^{(m-1)}}) - \omega_{y^{(m)}}\big|_{RE}\right)\Gamma_0 - \frac{1}{F_{y^{(n-1)}}}D_0\Gamma_0\right\}, \quad k \in \mathbb{N}_0,$$

where RE represents the reduced equation and Γ, Γ_0 are the linearised symmetry conditions of the governing and reduced equations respectively. Furthermore, the point-equivalent symmetry generators $X = Q\partial_y$ for the IVP are obtained by solving the system

$$\Big(D^m Q + c_1(D^{m-1}Q) + \cdots + c_{m-1}(DQ) + c_m Q\Big)\Big|_0 = 0$$

and

$$D^k\Gamma\big|_0 = 0 \quad \forall k \in \mathbb{N}_0.$$

Theorem 3.5. *For the initial-value problem given by* (24) *subject to*

$$y(0) = 0$$

the restriction of the linearised symmetry condition of the governing equation to solutions of the reduced equation provides the following relationships:

$$\Gamma|_{y=F} = -\frac{1}{F_{y^{(n-1)}}} D_0\Gamma_0 - (\omega_y)|_{y=F}\,\Gamma_0.$$

and

$$\left(D^k\Gamma\right)\big|_{y=F} = D_0^k\left\{-\frac{1}{F_{y^{(n-1)}}} D_0\Gamma_0 - (\omega_y)|_{y=F}\,\Gamma_0\right\}$$

where RE represents the reduced equation and Γ, Γ_0 are the linearised symmetry conditions of the governing and reduced equations respectively. Furthermore, the point-equivalent symmetry generators $X = Q\partial_y$ for the IVP are obtained by solving the system

$$(Q)|_0 = 0 \quad \textit{and } D^k\Gamma\big|_0 = 0 \quad \forall k \in \mathbb{N}_0.$$

Clearly, we can take any of the c_is of Theorem 3.3 or Theorem 3.4 to be zero and thus create a wealth of IVPs whose symmetry generators can, in principle, be found. Specifically, if $c_i = 0\;\forall\, i$ in Theorem 3.3, we have the IVP discussed by Hydon in (reference); similarly, if $c_i = 0\;\forall\, i$ in Theorem 3.4, we have the IVP discussed in the main analysis of this paper.

4 Examples

The third-order differential equation

$$y''' = xyy'' + 2yy' + xy'^2 \tag{28}$$

has a one-dimensional Lie algebra of point-equivalent symmetry generators spanned by

$$X_1 = (y + \frac{1}{2}xy')\partial_y.$$

Since (28) has one symmetry generator, we can only expect to reduce the order of the equation by one and then perform symmetry analysis on the resulting second-order equation as usual. To find the symmetries of the set of solutions that satisfy (28) subject to $y^{'}(0) = 0$, we are required to solve the determining system given in Theorem 3.2, where

$$Q(x, y, y') = \sum_{k=0}^{\infty} \Big(\eta_k(y) - y'\xi_k(y)\Big)x^k$$

and $\Gamma = 0$, where

$$\Gamma(x, y, y', y'') = D^3Q - (xy)D^2Q - 2(y + xy')DQ - (xy'' + 2y')Q.$$

From the Frobenius analysis presented earlier, we know that $Q|_0$ is, in fact, of the form

$$Q(x, y, \alpha, \gamma) = \sum_{k=0}^{\infty} \eta_k(y)x^k - \sum_{k=0}^{\infty}\sum_{r=0}^{\infty} w_r(\alpha, \gamma)\xi_k(y)x^{r+k}.$$

Specifically, for (28),

$$w_0 = \gamma, \quad w_2 = \frac{1}{2}\alpha\gamma \quad \text{etc.}, \qquad w_{2k+1} = 0 \; \forall \, k \in \mathbb{N}_0.$$

Hence, the invariance of the initial condition $y'(0) = 0$ under the symmetries of the IVP is given by

$$(DQ)|_0 = \eta_1 - \gamma\xi_1 = 0$$

which tells us that

$$\eta_1 = 0, \qquad \xi_1 = 0.$$

Collecting terms by powers of γ in the next determining equation gives

$$(\Gamma)|_0 = -3\eta_1'\gamma^2 + \gamma(-\xi_1\alpha + 3\eta_1' - 6\xi_3) + 6\eta_3 - 2\eta_1\alpha = 0,$$

which provides us with $\eta_3 = \xi_3 = 0$. The following determining equation $(D\Gamma)|_0 = 0$ splits to give

$$\begin{aligned} -3\xi_0'' &= 0, \\ 3\eta_0'' - 6\xi_0'\alpha - 12\xi_2' &= 0, \\ -6\xi_2\alpha - 24\xi_4 + 12\eta_2' - 3\eta_0 &= 0, \\ 24\eta_4 - 6\eta_2\alpha &= 0, \end{aligned}$$

from which

$$\xi_0 = c_2\alpha + c_1. \tag{29}$$

Continuing in this fashion (with the aid of computer algebra) we see that the remaining determining equations in fact constrain (29) and that the general solution of the determining equations is

$$\xi_0 = c_1, \qquad \xi_2 = \frac{1}{2}c_3, \qquad \eta_0 = c_3\alpha$$

with

$$\xi_1 = \eta_1 = \eta_2 = 0 \;\text{ and }\; \eta_k = \xi_k = 0 \;\; \forall\, k \geq 3.$$

We can then reconstruct the characteristic $Q(x, y, y')$ from (22) to give

$$Q(x, y, y') = \frac{c_1}{x} + \frac{1}{2}c_3 x + c_3 y$$

such that the point-equivalent symmetry generators are given by

$$X_1 = (y + \frac{1}{2}xy')\partial_y, \qquad X_2 = \frac{y'}{x}\partial_y.$$

Whilst X_1 is a symmetry of the governing equation, X_2 is a symmetry only of the reduced equation; furthermore it is of the structure we specifically altered the method of Hydon to accommodate. We can now use X_2 to find the reduced equation as follows. X_2 has differential invariants

$$r(x, y) = y, \quad v(x, y, y') = \frac{x}{y'}. \tag{30}$$

...

The second example considers the third-order ODE

$$y''' = \frac{1}{y^2} \tag{31}$$

subject to the linear combination of initial conditions

$$y''(0) + ay'(0) + by(0) = 0. \tag{32}$$

The governing equation (31) has symmetries generated by

...

Using the rifsimp (reference) package in MAPLE, a solution was found subject to the constraints

$$a^2 - 2b = 0, \quad a \neq 0,$$

the solution being

$$\eta_0 = -\frac{c_1 y}{a}, \quad \eta_1 = c_1 y \quad \xi_1 = -\frac{c_1}{a}, \quad \eta_2 = c_1$$

and all other η_k, ξ_k zero. This provides us with the point-equivalent symmetry generator

$$X_1 = x\left(x - \frac{1}{a}\right)\partial_x + y\left(x - \frac{1}{a}\right)\partial_y.$$

Note that if $a = b = 0$, i.e. we have the IVP given by (31) subject to $y''(0) = 0$, we have

$$\tilde{X}_1 = x\partial_x + y\partial_y.$$

...

5 Further work

The method presented in this paper allows a user to solve any given regular IVP, although the practical application of the determining systems may not be straightforward. Clearly, we would like to be able to extend this method to include all possible initial conditions, most notably for some governing equation subject to

$$y(0) = 0.$$

Furthermore, the reliance of the analysis on Taylor series means that it is only valid for *regular* ODEs (i.e. ODEs that are regular in the argument of the initial condition). If it were possible to amend the method to incorporate ODEs with regular singularities then the scope of the method would be considerably more appealing.

Building on the work concerning linear combinations of initial conditions, it would be useful if a user could know *a priori* if a particular combination of initial conditions is guaranteed to produce at least one new point-equivalent symmetry generator. Recently, "consistency" conditions have been introduced that allow a user to determine what boundary conditions are compatible with certain properties of a given differential equation. In a similar fashion, to be able to choose the constants c_i such that the associated linear combination of initial conditions admitted new symmetries would be particularly useful.

One of the many appealing features of symmetry analysis lies in the ability to follow a relatively simple algorithm to calculate symmetry generators. Although possible by hand, such calculations grow exponentially difficult according to the order of the differential equation in question; as a result, many computer algebra packages have been developed to calculate symmetry generators. The algorithm presented here has been converted into a MAPLE program to calculate the point-equivalent symmetry generators of third-order ODEs subject to $y'(0) = 0$. We aim to develop further computer algebra programs for calculating the symmetries of further IVPs and to apply this to various examples.

Although the extension of the method to higher-order ODEs was obvious, the extension to initial- and boundary-value problems for partial differential equations (PDEs) is yet to be investigated. Much of the recent research effort into I- and BVPs of PDEs has concentrated on higher-order, nonclassical symmetry methods and the categorisation of problems therein - what could be termed a *reactive* technique. Any possible extension of the method presented here to PDEs could provide a very useful, *proactive* technique whose benefits are obvious.

Obviously, if the method were to be successfully extended in scope to consider PDEs, the consideration of linear combinations of initial conditions would remain an integral part of the analysis. As such, to determine whether certain linear combinations of initial conditions guaranteed at least one new point-equivalent symmetry generator would also remain a goal for PDEs.

6 References